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ALMOST INVARIANT HALF-SPACES FOR OPERATORS ON HILBERT SPACE

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Abstract

The theory of almost invariant half-spaces for operators on Banach spaces was begun recently and is now under active development. Much less attention has been given to almost invariant half-spaces for operators on Hilbert space, where some techniques and results are available that are not present in the more general context of Banach spaces. In this note, we begin such a study. Our much simpler and shorter proofs of the main theorems have important consequences for the matricial structure of arbitrary operators on Hilbert space.

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1. Introduction

Two recent papers, [1, 5], began a study of almost invariant half-spaces (see Definition 1.1 below) for operators on infinite-dimensional complex Banach spaces. These papers were swiftly followed by several other contributions to this circle of ideas (namely, [2–4, 6, 7]), so this area of study is at present being vigorously developed. But, to the authors' knowledge, no article has appeared discussing these results solely in the context of Hilbert space. In this note we begin such a study and our proofs below (in the context of Hilbert space) of the main theorems of [1, 5] are much simpler and more transparent, which makes it possible to derive consequences of these theorems for operators on Hilbert space (for example, Theorem 3.1) that are not available for operators on more general spaces. This note is self contained and should be easily readable by researchers in the area.

Throughout, $\mathcal{H}$ will always denote a separable, infinite-dimensional, complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. We write, as

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usual, $\sigma(T)$ (respectively $\sigma_p(T)$, $\sigma_e(T)$) for the spectrum (respectively, point spectrum, essential spectrum) of an operator $T \in \mathcal{L}(\mathcal{H})$. The outer boundary of $\sigma(T)$, that is, the boundary of the unbounded component of $\mathbb{C}\backslash\sigma(T)$, is denoted by $\partial_0\sigma(T)$. We also denote the kernel of an operator T by $\mathcal{K}(T)$ and the closure of its range by $\mathcal{R}(T)$. As usual, we write $\mathbb{C}1_{\mathcal{H}}$ for the set of all scalar multiples of the identity operator $1_{\mathcal{H}}$.

Definition 1.1 [1]. A subspace (that is, a closed linear manifold) $\mathcal{M}$ in $\mathcal{H}$ such that $\dim \mathcal{M} = \dim \mathcal{M}^{\perp} = \aleph_0$ is called a *half-space* of $\mathcal{H}$. A half-space $\mathcal{M} \subset \mathcal{H}$ is said to be *almost invariant* for an operator $T \in \mathcal{L}(\mathcal{H})$ if there exists a finite-rank operator $F \in \mathcal{L}(\mathcal{H})$ such that $(T + F)\mathcal{M} \subset \mathcal{M}$.

Remark 1.2. It was noted in [1] that the condition in the definition above is easily seen to be equivalent to two other conditions:

(A) there exists a finite-dimensional subspace $\mathcal{E}$ of $\mathcal{H}$ such that $T\mathcal{M} \subset \mathcal{M} + \mathcal{E}$; and

(B) the 2×2 matrix of T with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$,

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{1.1}$$

has the property that $T_{21} : \mathcal{M} \to \mathcal{M}^{\perp}$ has finite rank.

In any case, if $\mathcal{M}$ is an almost invariant half-space for T, the *defect* of $\mathcal{M}$ is defined to be the rank of T_{21}, which is obviously also the minimum rank of an operator $F \in \mathcal{L}(\mathcal{H})$ such that $(T + F)\mathcal{M} \subset \mathcal{M}$ and the minimum dimension of a subspace $\mathcal{E} \subset \mathcal{H}$ such that $T\mathcal{M} \subset \mathcal{M} + \mathcal{E}$.

Remark 1.3. To shorten the exposition, write T has (AIHS) (respectively, (AIHSD$_n$)) to indicate that T has an almost invariant half-space (respectively, almost invariant half-space with defect n). If the defect of the (AIHS) for T is 0 or 1, we write that T has (AIHSD$_{0,1}$). It was also noted in [1] that the property of an operator T in $\mathcal{L}(\mathcal{H})$ having (AIHSD$_n$), where $n \in \mathbb{N}_0$, is preserved by the mappings $T \to T^*$ and $T \to \alpha T + \beta 1_{\mathcal{H}}$, where $\alpha \neq 0$ and β are any scalars.

The main theorem of the theory of (AIHS), moved to the context of Hilbert space, is the following statement.

Theorem 1.4 [1, 5, 6]. *Every operator T in $\mathcal{L}(\mathcal{H})$ has* (AIHSD$_{0,1}$).

It has been known for a long time that there are operators in $\mathcal{L}(\mathcal{H})$ which have no invariant half-space because all of their proper invariant subspaces are finite dimensional. One such operator is the backward Donoghue shift defined on an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ by $De_1 = 0$ and $De_n = (1/2^n)e_{n-1}$ for $n \in \mathbb{N}\backslash\{1\}$.

Thus, Theorem 1.4 is the best possible theorem obtainable that applies to all operators in $\mathcal{L}(\mathcal{H})$. It is therefore amazing that it has taken more than 80 years of interest in invariant subspaces for operators on Hilbert space for it to be found, and it is a significant achievement by the authors of [1, 5, 6] to originate techniques that led to its proof and to prove it.

2. Preliminary constructions

We will give a self-contained proof of Theorem 1.4, using only elementary Hilbert space techniques and the beautiful, ground-breaking construction from [1, 5]. But first comes the theorem that contains the essence of what the new construction from [1, 5] proves.

Theorem 2.1 [1, 5]. *Let T be an arbitrary operator in $\mathcal{L}(\mathcal{H})$ with the property that there is a point λ in $\partial_0(\sigma(T))$ such that $\lambda \notin \sigma_p(T) \cap \sigma_p(T^*)$ (which forces λ to belong to $\partial_0\sigma_e(T)$). Then T has* (AIHSD$_{0,1}$).

Proof. By Remark 1.3, we may translate T so that λ becomes 0. Furthermore, by exchanging T^* and T if necessary, we may suppose that $0 \notin \sigma_p(T)$. The argument now runs somewhat like those in [1, 5] except that our proof uses no entire functions. Choose a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ in the unbounded component of $\mathbb{C}\backslash\sigma(T)$ with $\lambda_n \to 0$ and observe that $\|(T - \lambda_n 1_{\mathcal{H}})^{-1}\| \to +\infty$. Hence, by the uniform boundedness principle, there exist a vector e and a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ of $\{\lambda_n\}_{n\in\mathbb{N}}$ (which we rename $\{\lambda_n\}_{n\in\mathbb{N}}$) such that $\|(T - \lambda_n 1_{\mathcal{H}})^{-1}e\| \to +\infty$. Consider the sequence of unit vectors

$$h_n = \alpha_n(T - \lambda_n 1_{\mathcal{H}})^{-1}e, \quad n \in \mathbb{N}, \tag{2.1}$$

where $\alpha_n = \|(T - \lambda_n 1_{\mathcal{H}})^{-1}e\|^{-1}$. Note that $\alpha_n \to 0$ and choose a subsequence $\{h_{n_j}\}_{j\in\mathbb{N}}$ (which we rename as $\{h_n\}_{n\in\mathbb{N}}$) that converges weakly to a vector h_0. Then (2.1) becomes

$$Th_n = \lambda_n h_n + \alpha_n e, \quad n \in \mathbb{N}. \tag{2.2}$$

The right-hand side of (2.2) tends to 0 as $n \to +\infty$ and the left-hand side converges weakly to Th_0. Thus, $Th_0 = 0$ and, since $0 \notin \sigma_p(T)$, $h_0 = 0$ and $h_n \xrightarrow{\text{w}} 0$. Next, let $\{e_k\}_{k\in\mathbb{N}}$ be any orthonormal basis for $\mathcal{H}$ and choose by induction a subsequence $\{h_{n_k}\}_{k\in\mathbb{N}}$ of $\{h_n\}_{n\in\mathbb{N}}$ such that $|\langle h_{n_j}, h_{n_m}\rangle| < 1/4^{j+m}$ for $j, m \in \mathbb{N}$, $j \neq m$. Define a linear transformation $S : \mathcal{H} \to \bigvee_{k\in\mathbb{N}} h_{n_k}$ by the equation $S(\sum_{k\in\mathbb{N}} \beta_k e_k) = \sum_{k\in\mathbb{N}} \beta_k h_{n_k}$ for every finitely nonzero square summable sequence $\{\beta_k\}_{k\in\mathbb{N}}$. The inequality above together with an easy calculation shows that whenever $\sum_{k\in\mathbb{N}} |\beta_k|^2 = 1$,

$$\frac{43}{45} \le \Big\| \sum_{k\in\mathbb{N}} \beta_k h_{n_k} \Big\|^2 \le \frac{47}{45},$$

and thus that S extends by continuity to a bounded invertible operator from $\mathcal{H}$ onto $\bigvee_{k\in\mathbb{N}}\{h_{n_k}\}$. Obviously, $\mathcal{M} = \bigvee_{k\in\mathbb{N}}\{h_{n_{2k}}\}$ and $\mathcal{N} = \bigvee_{k\in\mathbb{N}}\{h_{n_{2k-1}}\}$ have trivial intersection and thus $\mathcal{M}$ is a half-space of $\mathcal{H}$. The fact that $\mathcal{M}$ is an (AIHSD$_{0,1}$) follows immediately from (2.2). □

The following well-known definition and the easy proposition to follow will be used in our proof of Theorem 1.4.

Definition 2.2. A subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be a *semi-invariant subspace* for an operator A in $\mathcal{L}(\mathcal{H})$ if there exist invariant subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ for A with $\mathcal{N}_2 \subset \mathcal{N}_1$ such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$.

The best way to think about semi-invariant subspaces is to note that, with the above notation, $\mathcal{H} = \mathcal{N}_2 \oplus (\mathcal{N}_1 \ominus \mathcal{N}_2) \oplus \mathcal{N}_1^{\perp}$, the 3×3 operator matrix for A relative to this decomposition is in upper triangular form and the $(2, 2)$ entry of the matrix is the compression $P_{\mathcal{M}}A|_{\mathcal{M}}$ of A to $\mathcal{M}$, where $P_{\mathcal{M}}$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. In other words,

$$A = \begin{pmatrix} A|_{\mathcal{N}_2} & * & * \\ 0 & P_{\mathcal{M}}A|_{\mathcal{M}} & * \\ 0 & 0 & P_{\mathcal{N}_1^{\perp}}A|_{\mathcal{N}_1^{\perp}} \end{pmatrix}.$$

Proposition 2.3. *If $\mathcal{M}$ is an infinite-dimensional semi-invariant subspace for an operator A in $\mathcal{L}(\mathcal{H})$ and the compression of A to $\mathcal{M}$ has* (AIHSD$_{0,1}$), *then A has* (AIHSD$_{0,1}$).

Proof. By hypothesis, there exist half-spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ of $\mathcal{M}$ and a subspace $\mathcal{R} \subset \mathcal{M}_2$ of dimension 0 or 1 such that $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}$ and $(P_{\mathcal{M}}A|_{\mathcal{M}})\mathcal{M}_1 \subset \mathcal{M}_1 + \mathcal{R}$. An easy calculation then shows that $\mathcal{N}_2 \oplus \mathcal{M}_1$ is an (AIHSD$_{0,1}$) for A. □

Note also that an invariant subspace $\mathcal{K}$ of A is also a semi-invariant subspace for A because one can write $\mathcal{K} = \mathcal{K} \ominus (0)$. Thus, the proposition just proved applies equally well to a restriction of A to an invariant subspace $\mathcal{K}$.

3. The proofs

Proof of Theorem 1.4. For brevity and clarity the proof is split into cases.

Case I. $\partial_0\sigma(T)$ is an infinite set.

By Theorem 2.1, if there exists $\lambda_0 \in \partial_0\sigma(T)$ such that $\lambda_0 \notin \sigma_p(T) \cap \sigma_p(T^*)$, then T has (AIHSD$_{0,1}$). So we may suppose that there exist a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of distinct nonzero points of $\partial_0\sigma(T)$ converging to 0 and a corresponding sequence $\{v_n\}_{n\in\mathbb{N}}$ of eigenvectors such that $Tv_n = \lambda_n v_n$ for $n \in \mathbb{N}$. This means the invariant subspace $\mathcal{M} = \vee_{n\in\mathbb{N}}\{v_n\}$ for T is infinite dimensional (since eigenvectors corresponding to different eigenvalues are linearly independent). Moreover, $0 \in \partial_0\sigma_e(T|_{\mathcal{M}})$ and $T|_{\mathcal{M}}$ maps $\mathcal{M}$ to a dense linear manifold of itself. Consequently, $\mathcal{K}((T|_{\mathcal{M}})^*) = (0)$. By Theorem 2.1 and Proposition 2.3, $T|_{\mathcal{M}}$ and T have (AIHSD$_{0,1}$).

Case II. $\partial_0\sigma(T)$ is a finite set.

This obviously implies that $\sigma(T)$ is a finite set, one point at least of which must belong to $\sigma_e(T)$. By Remark 1.3, we may suppose that this point is 0. The Riesz idempotent E associated with the isolated point 0 in $\sigma(T)$ must have range an infinite-dimensional subspace $\mathcal{E}$ that is invariant under T. By Proposition 2.3, we may exchange T for $T|_{\mathcal{E}}$. In other words, we may suppose that T is quasinilpotent.

Case III. T is quasinilpotent.

If T is nilpotent, that T has (AIHSD$_0$) is trivial since $\mathcal{K}(T)$ is infinite dimensional, so we may suppose that T is not nilpotent. Moreover if $0 \notin \sigma_p(T) \cap \sigma_p(T^*)$, then the result follows from Theorem 2.1. Thus we are reduced to the case in which both $\mathcal{K}(T)$ and $\mathcal{K}(T^*)$ are nonzero and finite dimensional. Observe next that if $\mathcal{M}$ is any infinite dimensional invariant subspace of T^*, then $0 \in \partial_0\sigma_e(T^*|_{\mathcal{M}})$. Thus it sufficies to find an infinite dimensional invariant subspace $\mathcal{M}$ for T^* such that $\mathcal{R}(T^*|_{\mathcal{M}}) = \mathcal{M}$, which we now do. If for every $x \in \mathcal{H}$, some nonzero polynomial p_x satisfies $p_x(T)x = 0$, then by Kaplansky's Lemma, T is an algebraic operator and hence nilpotent, which case has already been considered. Thus there exists $x \in \mathcal{H}$ such that the set $\{T^n x\}_{n\in\mathbb{N}}$ is linearly independent. Let us write $C_x = \vee_{n=0}^{\infty}\{T^n x\}$ which is an infinite dimensional cyclic invariant subspace for T. By Proposition 2.3, without loss of generality we may suppose that $C_x = \mathcal{H}$, so that x is a cyclic vector for T. The only possibilities for $\mathcal{K}(T^*)$ are thus $\mathcal{K}(T^*) = (0)$ or $\dim\mathcal{K}(T^*) = 1$. In the first case an application of Theorem 2.1 completes the proof, so we arrive at the case that $\dim\mathcal{K}(T^*) = 1$. Observe that it follows easily that $\dim\mathcal{K}(T^{*n}) \le n$ for all $n \in \mathbb{N}$. Moreover, if there exists $k \in \mathbb{N}$ such that $\mathcal{K}(T^{*k}) = \mathcal{K}(T^{*k+1})$, then $\mathcal{R}(T^k) = \mathcal{R}(T^{k+1})$, which gives $\mathcal{K}((T|_{\mathcal{R}(T^k)})^*) = (0)$ and, thus, by Proposition 2.3 and Theorem 2.1, T has (AIHSD$_{0,1}$). Therefore, we arrive at the case in which $\dim(\mathcal{K}(T^{*k+1}) \ominus \mathcal{K}(T^{*k})) = 1$ for every $k \in \mathbb{N}$ and consequently we obtain an invariant subspace $\mathcal{M}$ for T^*, namely

$$\mathcal{M} = \vee_{n=1}^{\infty}\mathcal{K}(T^{*n}) = \mathcal{K}(T^*) \oplus (\mathcal{K}(T^{*2}) \ominus \mathcal{K}(T^*)) \oplus (\mathcal{K}(T^{*3}) \ominus \mathcal{K}(T^{*2})) \oplus \cdots.$$

Therefore, we may choose an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ for $\mathcal{H}$ by choosing

$$e_1 \in \mathcal{K}(T^*), \quad e_{n+1} \in \mathcal{K}(T^{*n+1}) \ominus \mathcal{K}(T^{*n}) \quad \text{for } n \in \mathbb{N}$$

and write the matrix for $T^*|_{\mathcal{M}}$ with respect to this orthonormal basis as

$$T^*|_{\mathcal{M}} = \begin{pmatrix} 0 & t_{12} & t_{13} & t_{14} & \cdots \\ 0 & 0 & t_{23} & t_{24} & \cdots \\ 0 & 0 & 0 & t_{34} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{3.1}$$

Observe that for every $j \in \mathbb{N}$, the entry $t_{j,j+1} \neq 0$, because otherwise, if j_0 is the smallest positive integer such that $t_{j_0,j_0+1} = 0$, then $\dim\mathcal{K}(T^{*j_0}) = j_0 + 1$, which contradicts the already established fact that $\dim\mathcal{K}(T^{*n}) = n$, $n \in \mathbb{N}$. The proof of the theorem is completed by observing now that the range of $T^*|_{\mathcal{M}}$ is dense in $\mathcal{M}$. Indeed, if the Gram–Schmidt procedure is applied to all but the first column of (3.1), an orthonormal basis for $\mathcal{M}$ results. Thus $\mathcal{R}(T^*|_{\mathcal{M}}) = \mathcal{M}$ and applying Proposition 2.3 and Theorem 2.1 completes the proof.

The following corollary of Theorem 1.4 for operators on Hilbert space seems to be new.

Theorem 3.1. *Every operator in* $\mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ *has* (AIHSD$_1$).

We note at once that this also is the best possible theorem since scalar multiples of $1_{\mathcal{H}}$ cannot have (AIHSD$_1$). We show now that Theorem 3.1 follows from Theorem 1.4. It obviously suffices to obtain the following proposition.

Proposition 3.2. *Suppose that $T \in \mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$ and has the property that $\mathcal{M} \subset \mathcal{H}$ is an invariant half-space for T. Then T also has* (AIHSD$_1$).

The proof of Proposition 3.2 uses the following lemmas.

Lemma 3.3. *The property of an operator T in $\mathcal{L}(\mathcal{H})$ having* (AIHSD$_n$) *is preserved under similarity transformations $T \to STS^{-1}$.*

Proof. As noted in Definition 1.1, the hypothesis implies that there exist a half-space $\mathcal{M}$ and an operator F of minimum rank n such that $(T+F)\mathcal{M} \subset \mathcal{M}$. Let S be any invertible operator in $\mathcal{L}(\mathcal{H})$ and note that

$$S(T+F)S^{-1}(S\mathcal{M}) = S(T+F)\mathcal{M} \subset S\mathcal{M}$$

and SFS^{-1} has rank n. Thus, $S\mathcal{M}$ is an almost invariant half-space for STS^{-1} with defect n. □

Lemma 3.4. *Suppose that $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{M} \subset \mathcal{H}$ is a half-space such that the matrix for T relative to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$ has the form*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}. \tag{3.2}$$

Then

$$T \notin \mathbb{C}1_{\mathcal{H}} \leftrightarrow [(T_{12} \neq 0) \vee (T_{11} \notin \mathbb{C}1_{\mathcal{H}}) \vee (T_{22} \notin \mathbb{C}1_{\mathcal{H}}) \vee (T_{11} \neq T_{22})]. \tag{3.3}$$

Proof. After taking the denial of (3.3) and applying the distributive law to the right-hand side of the resulting proposition, the result becomes obvious. □

Lemma 3.5. *If $T \in \mathcal{L}(\mathcal{H})\backslash\mathbb{C}1_{\mathcal{H}}$, then there exist orthonormal vectors g and h in $\mathcal{H}$ such that $\langle Tg, h\rangle \neq 0$.*

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$. If the matrix for T relative to the orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ is not diagonal, there is nothing to prove. Thus, suppose that this matrix is diagonal. Then, of necessity, there exist vectors e_j and e_k in this basis such that $Te_j = \alpha e_j$ and $Te_k = \beta e_k$ with $\alpha \neq \beta$. Now define $g = (e_j + e_k)/\sqrt{2}$ and $h = (e_j - e_k)/\sqrt{2}$. Then $\{g, h\}$ is an orthonormal set and $\langle Tg, h\rangle = (\alpha - \beta)/2 \neq 0$, so the lemma is proved. □

Proof of Proposition 3.2. We are given $T \in \mathcal{L}(\mathcal{H})$ that is not a scalar multiple of $1_{\mathcal{H}}$ and has an invariant half-space $\mathcal{M}$, so the matrix for T as in (1.1) has the form (3.2). We now apply Lemma 3.4 and consider cases.

Case I: $T_{22} \notin \mathbb{C}1_{\mathcal{H}}$.

In this case it follows from Lemma 3.5 that there exist orthonormal vectors e_0 and f_0 in $\mathcal{M}^{\perp}$ such that $\langle T_{22}e_0, f_0\rangle$ $(= \langle Te_0, f_0\rangle) \neq 0$. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{M}$ and extend the pair $\{e_0, f_0\}$ to an orthonormal basis $\{e_0, \{f_n\}_{n\in\mathbb{N}_0}\}$ for $\mathcal{M}^{\perp}$. Obviously, $\{e_n\}_{n\in\mathbb{N}_0} \cup \{f_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis for $\mathcal{H}$. Define now a subspace $\mathcal{N}$

of $\mathcal{H}$ by $\mathcal{N} = \bigvee_{n\in\mathbb{N}_0}\{e_n\}$ and note that $\mathcal{N}^\perp$ satisfies the formula $\mathcal{N}^\perp = \bigvee_{n\in\mathbb{N}_0}\{f_n\}$. If the 2×2 matrix for T relative to the decomposition $\mathcal{H} = \mathcal{N}\oplus\mathcal{N}^\perp$ is (T'_{ij}), then elementary calculations show that T'_{21} has rank one, which completes the proof in Case I.

Case II. There exists $\alpha\in\mathbb{C}$ such that $T_{22} = \alpha 1_{\mathcal{H}}$ and $T_{12}\neq 0$.

In this case we compute the similarity

$$STS^{-1} = \begin{pmatrix} 1_{\mathcal{H}} & 0 \\ X & 1_{\mathcal{H}} \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} \\ 0 & \alpha I_{\mathcal{H}} \end{pmatrix}\begin{pmatrix} 1_{\mathcal{H}} & 0 \\ -X & 1_{\mathcal{H}} \end{pmatrix} = \begin{pmatrix} T_{11} - T_{12}X & T_{12} \\ * & \alpha 1_{\mathcal{H}} + XT_{12} \end{pmatrix},$$

where $X : \mathcal{M}\to\mathcal{M}^\perp$ is an arbitrary bounded operator. Since $T_{12}\neq 0$, it is trivial to choose X of rank one such that $\alpha 1_{\mathcal{H}} + XT_{12}\notin\mathbb{C}1_{\mathcal{H}}$. By Lemma 3.3, it suffices to show that STS^{-1} has (AIHSD$_1$), and now we are back in Case I with STS^{-1} in place of T. Thus, the argument in Case II is complete.

Case III. $T_{22} = \alpha 1_{\mathcal{H}}$, $T_{12} = 0$ and $T_{11}\notin\mathbb{C}1_{\mathcal{H}}$.

Proceeding as in Case I, there exist orthonormal vectors e_0 and f_0 in $\mathcal{M}$ such that $\langle T_{11}e_0, f_0\rangle$ $(= \langle Te_0, f_0\rangle)\neq 0$. Define $\mathcal{K} = \mathcal{M}\ominus\vee\{f_0\}$ so that $\mathcal{K}^\perp = \mathcal{M}^\perp\oplus\vee\{f_0\}$. If the 2×2 matrix for T relative to the decomposition $\mathcal{H} = \mathcal{K}\oplus\mathcal{K}^\perp$ is (T''_{ij}), then trivial computations show that T''_{21} has rank one and thus the proof in this case is complete.

Case IV. $T_{22} = \beta 1_{\mathcal{H}}$, $T_{12} = 0$, $T_{11} = \gamma 1_{\mathcal{H}}$ and $\beta\neq\gamma$.

Let $\{e_n\}_{n\in\mathbb{N}}$ and $\{f_n\}_{n\in\mathbb{N}}$ be orthonormal bases for $\mathcal{M}$ and $\mathcal{M}^\perp$, respectively, and define $g_1 = (e_1 + f_1)/\sqrt{2}$ and $h_1 = (e_1 - f_1)/\sqrt{2}$. A trivial calculation shows that $\langle Tg_1, h_1\rangle = (\gamma-\beta)/2\neq 0$. Define the subspace $\mathcal{L}$ of $\mathcal{H}$ as $\mathcal{L} = \vee\{g_1, \{e_n\}_{n\in\mathbb{N}\setminus\{1\}}\}$. Then $\mathcal{L}^\perp = \vee\{h_1, \{f_n\}_{n\in\mathbb{N}\setminus\{1\}}\}$ and, in fact, these two sets of vectors are orthonormal bases for $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively. Easy calculations now show that if (T'''_{ij}) is the 2×2 matrix of T relative to the decomposition $\mathcal{H} = \mathcal{L}\oplus\mathcal{L}^\perp$, then T'''_{21} has rank one. This completes the proof of Theorem 3.1 and Proposition 3.2.

Remark 3.6. One might think that the steps in the proof of Proposition 3.2 could be reversed and thereby solve the invariant subspace problem for operators on Hilbert space. But the example of the Donoghue shift D given earlier shows that this is not possible.

Remark 3.7. In the sequel to this article 'Almost invariant half-spaces of operators on Hilbert space II: operator matrices', now in preparation, we explore consequences of our proofs of Theorems 1.4 and 2.1 and obtain several new theorems about the matricial structure of operators in $\mathcal{L}(\mathcal{H})$.

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